Paper

On q-Laplace transformation

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Abstract In this paper, we treat a q-Laplace transformation in $f(x)=\sum_{n=0}^{\infty}\frac{a_n}{(n!)_q}x^n$. In the first half, we state that the similar transformation, the shift transformation, the transformation in differantial and integral are satisfied. Especialy, in section 3, we treat a q-analogue of β-function and a q-Laplace transform in convolution $(f*g)(x)$. In second, we solve some linear ordinary differential equations of second order with constant coefficients by using an inverse q-Laplace transformation and obtain a q-Laplace transformation in product of an error function and an exponential function.

Keywords : q-Laplace transformation, convolution, differential equations, error function

1. NOTATIONS

Let q be a fixed number with $0<q<1$

First, we put $(a\ ;\ q)_n=\prod_{k=1}^{n}(1-q^{k-1}a)$ for $n\in \mathrm{N}$, $(a\ ;\ q)_0=1$ and $(a\ ;\ q)_\infty=\prod_{k=1}^{\infty}(1-q^{k-1}a)$.

Second, for $n\in \mathrm{N}$, we put $n_q=\sum_{k=1}^{n}q^{k-1}$ and $\infty_q=\sum_{k=1}^{\infty}q^{k-1}=\frac{1}{1-q}$. Futhermore, we put $(n!)_q=\prod_{k=1}^{n}k_q$ (where $(0!)_q=1$), $\binom{n}{k}_q=\frac{(n!)_q}{(k!)_q\{(n-k)!\}_q}$ ($k=0,\ 1,\ \cdots,\ n$), $[x]^n=\prod_{k=1}^{n}(q^{k-1}x)=q^{\frac{1}{2}n(n-1)}x^n$ and $(x+[y])^n=\prod_{k=1}^{n}(x+q^{k-1}y)$.

And these notations are rewritten $n_q=\frac{1-q^n}{1-q}$, $(n!)_q=\frac{(q\ ;\ q)_n}{(1-q)^n}=\frac{(q\ ;\ q)_\infty}{(q^{n+1}\ ;\ q)_\infty}(1-q)^{-n}$, $[-1]^k\binom{n}{k}_q=\frac{(q^{-n}\ ;\ q)_k}{(q\ ;\ q)_k}q^{nk}$ and $(x+[y])^n=x^n(-y/x\ ;\ q)_n=x^n\frac{(-y/x\ ;\ q)_\infty}{(-q^n y/x\ ;\ q)_\infty}$ ($x\neq 0$).

Then we generalize the above notations by replacing $n\in \mathrm{N}$ with $\alpha\in \mathrm{R}$. That is, we put $\alpha_q=\frac{1-q^\alpha}{1-q}$, $[-1]^k\binom{\alpha}{k}_q=\frac{(q^{-\alpha}\ ;\ q)_k}{(q\ ;\ q)_k}q^{\alpha k}$ and $(x+[y])^\alpha=x^\alpha\frac{(-y/x\ ;\ q)_\infty}{(-q^\alpha y/x\ ;\ q)_\infty}$. In (1), for $\alpha>0$, we find that the q-analouge of Γ-function is defined by $\Gamma_q(\alpha)=\frac{(q\ ;\ q)_\infty}{(q^\alpha\ ;\ q)_\infty}(1-q)^{1-\alpha}$. In fact, by easy calculations, we obtain $\Gamma_q(1)=1$ and $\Gamma_q(\alpha+1)=\alpha_q\Gamma_q(\alpha)$.

Next, we state two q-analogues of exeponential function e^x. In (1), we find $e_q(x)=\frac{1}{((1-q)x\ ;\ q)_\infty}$ ($|x|<\infty_q$) and $E_q(x)=(-(1-q)x\ ;\ q)_\infty$ ($|x|<\infty$). We know that these are rewritten $e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n!)_q}$ and $E_q(x)=\sum_{n=0}^{\infty}\frac{[x]^n}{(n!)_q}$

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by the q-binominal theorem.

(See (1).)
Theorem (q-biominal theorem).

(1.1) $$\sum_{n=0}^{\infty}\frac{(a\ ;\ q)_n}{(q\ ;\ q)_n}x^n=\frac{(ax\ ;\ q)_\infty}{(x\ ;\ q)_\infty}\quad (|x|<1),$$

(1.2) $$\sum_{n=0}^{\infty}\frac{1}{(q\ ;\ q)_n}x^n=\frac{1}{(x\ ;\ q)_\infty}\quad (|x|<1),$$

(1.3) $$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q\ ;\ q)_n}x^n=(x\ ;\ q)_\infty .$$

We remark that $(x+[y])^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}_q x^{\alpha-n}[y]^n$ is satisfied with $x\neq 0$ and $\left|\frac{y}{x}\right|<q^{-\alpha}$.

Finally, we state the q-differential and the Jackson's integral. In (1), the q-differential operator Δ_q is defined by $\Delta_q f(x)=\frac{f(x)-f(qx)}{(1-q)x}$, and Jackson's integral is defined by $\int_0^x f(t)\ d_qt=\sum_{n=0}^{\infty}(1-q)q^n x\ f(q^n x)$. Futermore we put $\tilde{\Delta}_q f(x)\ =\frac{f(x)-f(q^{-1}x)}{(1-q^{-1})x}$ and $\int_x^\infty f(t)\ d_qt=-\sum_{n=0}^{\infty}(1-q^{-1})q^{-n}x\ f(q^{-n}x)$.

2. DEFINITION

Let $f(x)$ be a function for $x\geq 0$. We define a q-Laplace transformation in $f(x)$ as

$$L_q[f(x)]=\frac{1}{s}\int_0^{\infty_q}E_q(-qx)f\left(\frac{x}{s}\right)d_qx .$$

Example 1. For x^α ($\alpha>-1$), we have

$$L_q[x^\alpha]=\frac{1}{s}\int_0^{\infty_q}E_q(-qx)\left(\frac{x}{s}\right)^\alpha d_qx=\frac{(q\ ;\ q)_\infty}{(1-q)^\alpha s^{\alpha+1}}\sum_{n=0}^{\infty}\frac{q^{(\alpha+1)n}}{(q\ ;\ q)_n}=\frac{\Gamma_q(\alpha+1)}{s^{\alpha+1}} .$$

We put $f(x)=\sum_{n=0}^{\infty}\frac{a_n}{(n!)_q}x^n$. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=r\ (<\infty)$, $f(x)$ is convergent in $[0,\ \infty_q/r)$. That is, if $s>r$, $f\left(\frac{x}{s}\right)$ is convergent in $[0,\ \infty_q]$. Then we have a following theorem.

Theorem 2. Suppose that a sequence $\{a_n\}$ is satisfied with $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=r\ (<\infty)$.

Then, a q-Laplace tarnsformation of $f(x)=\sum_{n=0}^{\infty}\frac{a_n}{(n!)_q}x^n$ ($0\leq x<\infty_q/r$) is given by $L_q[f(x)]=\sum_{n=0}^{\infty}\frac{a_n}{s^{n+1}}$.

Proof. From Example 1, we have

$$L_q[f(x)]=\sum_{n=0}^{\infty}\frac{a_n}{(n!)_q}L_q[x^n]=\sum_{n=0}^{\infty}\frac{a_n}{(n!)_q}\cdot\frac{\Gamma_q(n+1)}{s^{n+1}}=\sum_{n=0}^{\infty}\frac{a_n}{s^{n+1}} .$$

Example 2. For a q-exponential function $e_q(\lambda x)=\sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)_q}x^n$ $(0\le x<\infty_q/|\lambda|)$, we have

$$L_q[e_q(\lambda x)]=\sum_{n=0}^{\infty}\frac{\lambda^n}{s^{n+1}}=\frac{1}{s-\lambda} \quad (s>|\lambda|).$$

3. PROPERTIES

In this section, a sequence $\{a_n\}$ is satisfied with $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=r\ (<\infty)$. We put $f(x)=\sum_{n=0}^{\infty}\frac{a_n}{(n!)_q}x^n$ $(0\le x<\infty_q/r)$ and $F_q(s)=\sum_{n=0}^{\infty}\frac{a_n}{s^{n+1}}$ $(s>r)$. When there is a number N such that $a_n=0$ for $n\ge N$, we put $r=0$ and ∞_q/r is equal to infinity.

Then, we show the similar tansformation, the shift transformation, the transform in differantion and integration.

Proposition 1. (similar transformation)

$$(3.1)\quad L_q[f(\lambda x)]=\frac{1}{\lambda}F_q\left(\frac{s}{\lambda}\right) \quad (0\le x<\infty_q/|\lambda|r,\ s>|\lambda|r).$$

Proof. From $f(\lambda x)=\sum_{n=0}^{\infty}\frac{\lambda^n a_n}{(n!)_q}x^n$, we have

$$L_q[f(\lambda x)]=\sum_{n=0}^{\infty}\frac{a_n\lambda^n}{s^{n+1}}=\frac{1}{\lambda}F\left(\frac{s}{\lambda}\right).$$

Proposition 2.(shift transformation)

We put $R=\max\{r,\ |\lambda|\}$. Then the equation

$$(3.2)\quad L_q[e_q(\lambda x)f(x)]=F_q(s-[\lambda]) \quad (0\le x<\infty_q/R,\ s>R)$$

is satisfied, where we put $F_q(s-[\lambda])=\sum_{n=0}^{\infty}\frac{a_n}{(s-[\lambda])^{n+1}}$.

Proof. From $e_q(\lambda x)f(x)=\left\{\sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)_q}x^n\right\}\left\{\sum_{k=0}^{\infty}\frac{a_k}{(k!)_q}x^k\right\}=\sum_{n=0}^{\infty}\frac{x^n}{(n!)_q}\sum_{k=0}^{n}\binom{n}{k}_q\lambda^{n-k}a_k$,

we have

$$\begin{aligned}L_q[e_q(\lambda x)f(x)]&=\sum_{n=0}^{\infty}\frac{1}{s^{n+1}}\sum_{k=0}^{n}\binom{n}{n-k}_q\lambda^{n-k}a_k\\&=\sum_{k=0}^{\infty}\frac{a_k}{s^{k+1}}\sum_{n=0}^{\infty}\binom{n+k}{n}_q\left(\frac{\lambda}{s}\right)^n\\&=\sum_{k=0}^{\infty}\frac{a_k}{s^{k+1}}\sum_{n=0}^{\infty}\frac{(q^{k+1}\ ;\ q)_n}{(q\ ;\ q)_n}\left(\frac{\lambda}{s}\right)^n\\&=\sum_{k=0}^{\infty}\frac{a_k}{s^{k+1}(\lambda/s\ ;\ q)_{k+1}}\\&=\sum_{k=0}^{\infty}\frac{a_k}{(s-[\lambda])^{k+1}}\end{aligned}$$

$$= F_q(s-[\lambda]) .$$

Proposition 3.(transformation in differantion)

(3.3) $L_q[\Delta_q f(x)] = sF_q(s) - f(0) \quad (0 \le x < \infty_q / r \,, s > r)$,

(3.4) $L_q[xf(x)] = -\frac{1}{q}\tilde{\Delta}_q F_q(s) \quad (0 \le x < \infty_q / r \,, s > r)$.

Proof. (3.3) From $\Delta_q f(x) = \sum_{n=1}^{\infty} \frac{a_n}{\{(n-1)!\}_q} x^{n-1} = \sum_{n=0}^{\infty} \frac{a_{n+1}}{(n!)_q} x^n$, we have

$$L_q[\Delta_q f(x)] = \sum_{n=0}^{\infty} \frac{a_{n+1}}{s^{n+1}} = s\left\{ \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}} - \frac{a_0}{s} \right\} = sF_q(s) - f(0) .$$

(3.4) From $xf(x) = \sum_{n=0}^{\infty} \frac{a_n}{(n!)_q} x^{n+1} = \sum_{n=0}^{\infty} \frac{n_q a_{n-1}}{(n!)_q} x^n$, where we put a_{-1} as an arbitrary constant by $0_q = 0$, we have

$$L_q[xf(x)] = \sum_{n=0}^{\infty} \frac{n_q a_{n-1}}{s^{n+1}} = \sum_{n=0}^{\infty} \frac{(n+1)_q a_n}{s^{n+2}} .$$

On the other hand, we have

$$\tilde{\Delta}_q F_q(s) = \frac{F_q(s) - F_q(q^{-1}s)}{(1-q^{-1})s} = -q\sum_{n=0}^{\infty} \frac{(n+1)_q a_n}{s^{n+2}} .$$

Thus, (3.4) is obtained.

Proposition 4.(transform in integration)

(3.5) $L_q\left[\int_0^x f(t)\ d_q t\right] = \frac{1}{s} F_q(s) \quad (0 \le x < \infty_q / r \,, s > r)$,

(3.6) If $f(0) = 0$, $L_q\left[\frac{1}{x} f(x)\right] = q\int_s^{\infty} F_q(t)\ d_q t \quad (0 \le x < \infty_q / r \,, s > r)$.

proof. (3.5) From $\int_0^x f(t)\ d_q t = \sum_{n=0}^{\infty} \frac{a_n}{\{(n+1)!\}_q} x^{n+1} = \sum_{n=0}^{\infty} \frac{a_{n-1}}{(n!)_q} x^n$, where $a_{-1} = 0$, we have

$$L_q\left[\int_0^x f(t)\ d_q t\right] = \sum_{n=0}^{\infty} \frac{a_{n-1}}{s^{n+1}} = \frac{1}{s}\sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}} = \frac{1}{s} F_q(s) .$$

(3.6) From $f(0) = 0$, we have $a_0 = 0$. Thus we obtain $\frac{1}{x} f(x) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{\{(n+1)!\}_q} x^n$.

Therefore, we have $L_q\left[\frac{1}{x} f(x)\right] = \sum_{n=0}^{\infty} \frac{a_{n+1}}{(n+1)_q s^{n+1}} = \sum_{n=1}^{\infty} \frac{a_n}{n_q s^n}$.

On the other hand, we have

$$\int_s^{\infty} F_q(t)\ d_q t = -\sum_{n=0}^{\infty} (1-q^{-1}) q^{-n} s F_q(q^{-n}s) = \frac{1-q}{q} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{q^{kn} a_k}{s^k} = \frac{1-q}{q} \sum_{k=1}^{\infty} \frac{a_k}{s^k} \sum_{n=0}^{\infty} q^{kn} = \frac{1}{q} \sum_{k=1}^{\infty} \frac{a_k}{k_q s^k}$$

Thus, (3.6) is obtained.

4. CONVOLUTION

In this section, we treat a q-Laplace transformation in convolution.
First, we state a scaling of Jackson's integral.

Lenmma 1. $\int_0^x f(t)\ d_q t = x\int_0^1 f(xt)\ d_q t$

Proof. We have

$$\int_0^x f(t)\ d_q t = \sum_{n=0}^{\infty}(1-q)q^n x\ f(q^n x) = x\int_0^1 f(xt)\ d_q t\ .$$

Next, we state β -function and definition of convolution. In (1), a q-analogue of β -function is defined by

$$\beta_q(\alpha,\beta) = \int_0^1 t^{\alpha-1}\frac{(qt\ ;\ q)_\infty}{(q^\beta t\ ;\ q)_\infty}\ d_q t \quad (\alpha,\ \beta > 0\ ).$$

Indeed, by esay calculations, we can obtain $\beta_q(\alpha,\beta) = \dfrac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$. And a q-analogue of β -function is rewritten $\beta_q(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-[qt])^{\beta-1}\ d_q t$ by using our notations.

Then, we define a q-analogue of convolution of $f(x)$ and $g(x)$ as

$$(f*g)(x) = \int_0^x f(t)g(x-[qt])\ d_q t\ .$$

So, this definition is an extension in a q-analogue of β -function.

Suppose that sequences $\{a_n\},\ \{b_n\}$ are satisfied with $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = r_1,\ \lim_{n\to\infty}\left|\dfrac{b_{n+1}}{b_n}\right| = r_2$. We put $f(x) = \sum_{n=0}^{\infty}\dfrac{a_n}{(n!)_q}x^n$, $g(x) = \sum_{n=0}^{\infty}\dfrac{b_n}{(n!)_q}x^n$, $F_q(s) = \sum_{n=0}^{\infty}\dfrac{a_n}{s^{n+1}}$, and $G_q(s) = \sum_{n=0}^{\infty}\dfrac{b_n}{s^{n+1}}$. Then, equations $L_q[f(x)] = F_q(s)\quad (0 \le x < \infty_q/r_1, s > r_1)$, $L_q[g(x)] = G_q(s)\quad (0 \le x < \infty_q/r_2,\ s > r_2)$ are satisfied. Futhermore we put $g(\alpha-[qx]) = \sum_{n=0}^{\infty}\dfrac{b_n}{(n!)_q}(\alpha-[qx])^n$. we remark that $g(\alpha-[qx])$ is convergent in $x \ge 0$, if $0 \le \alpha < \infty_q/r_2$. Then we put $r = \max\{r_1,\ r_2\}$, we have the follwing Lemma for $0 \le x < \infty_q/r$.

Lemma 2. $(f*g)(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\dfrac{a_n\ b_k}{\{(n+k+1)!\}}\ x^{n+k+1}$

Proof. We have

$$\begin{aligned}(f*g)(x) &= \int_0^x f(t)g(x-[qt])\ d_q t \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_n b_k}{(n!)_q(k!)_q}\int_0^x t^n(x-[qt])^k\ d_q t \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_n b_k}{(n!)_q(k!)_q}x^{n+k+1}\int_0^1 t^n(1-[qt])^k\ d_q t \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_n b_k}{(n!)_q(k!)_q}x^{n+k+1}\cdot\frac{\Gamma_q(n+1)\Gamma_q(k+1)}{\Gamma_q(n+k+2)} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_n\ b_k}{\{(n+k+1)!\}}\ x^{n+k+1}\ .\end{aligned}$$

Finally, we state a q-Laprace transformation in convolution.

Proposition 5.(q-Laplace transformatiopn in convolution)

$$L_q[(f*g)(x)] = F_q(s)\ G_q(s) \quad (0 \le x < \infty_q / r,\ s > r)$$

Proof. From lemma 2, we have

$$L_q[(f*g)(x)] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{a_k\ b_l}{s^{k+l+2}} = \left(\sum_{k=0}^{\infty}\frac{a_k}{s^{k+1}}\right)\left(\sum_{l=0}^{\infty}\frac{b_l}{s^{l+1}}\right) = F_q(s)\ G_q(s).$$

5. DIFFERENTIAL EQUATIONS

For a sequence $\{a_n\}$, a function $f(x) = \sum_{n=0}^{\infty}\frac{a_n}{(n!)_q}x^n$ corresponds to a function $F_q(s) = \sum_{n=0}^{\infty}\frac{a_n}{s^{n+1}}$. So we put $L_q^{-1}[F_q(s)] = f(x)$. This is an inverse correspondence of q-Laplace transformation.

(DE1) $$\begin{cases}\Delta_q^2 f(x) - (\alpha+\beta)\Delta_q f(x) + \alpha\beta f(x) = 0 \\ f(0) = A,\ \Delta_q f(0) = B\end{cases}$$

We remark that both L_q and L_q^{-1} are cleary satisfied linearly. And, by easy caluculations, we can obtain

$$L_q[\Delta_q^2 f(x)] = s^2 F_q(x) + sf(0) - \Delta_q f(0).$$

Then, from (DE1), we have

$$F_q(s) = \frac{A(s-\alpha-\beta)+B}{(s-\alpha)(s-\beta)} = \frac{B-\beta A}{(s-\alpha)(\alpha-\beta)} + \frac{B-\alpha A}{(\beta-\alpha)(s-\beta)}.$$

Thus,

$$f(x) = \frac{B-\beta A}{(\alpha-\beta)}e_q(\alpha x) + \frac{B-\alpha A}{(\beta-\alpha)}e_q(\beta x)$$

is a solution of (DE1).

(DE2) $$\begin{cases}\Delta_q^2 f(x) - 2_q\alpha\ \Delta_q f(x) + q\alpha^2 f(x) = 0 \\ f(0) = A,\ \Delta_q f(0) = B\end{cases}$$

From (DE2), we have

$$F_q(s) = \frac{A(s-\alpha-q\alpha)+B}{(s-\alpha)(s-q\alpha)} = \frac{A}{s-\alpha} + \frac{B-\alpha A}{(s-[\alpha])^2}.$$

On the other hand, we have

$$-\frac{1}{q}\tilde{\Delta}_q\left(\frac{1}{s-\alpha}\right) = \frac{1}{(s-[\alpha])^2}.$$

Thus, from (3.4),

$$f(x) = Ae_q(\alpha x) + (B-\alpha A)xe_q(\alpha x)$$

is a solution of (DE2).

(DE3) $$\begin{cases}\Delta_q^2 f(x) + f(x) = 0 \\ f(0) = A,\ \Delta_q f(0) = B\end{cases}$$

For a sequence $a_n = \begin{cases}0, & n = 2m \\ (-1)^m, & n = 2m+1\end{cases}$ $(m = 0,\ 1,\ 2,\ \cdots)$, we put

$$Sin_q(x)=\sum_{n=0}^{\infty}\frac{a_n}{(n!)}x^n=\sum_{m=0}^{\infty}\frac{(-1)^m}{\{(2m+1)!\}_q}x^{2m+1}\quad(0\le x<\infty_q).$$

Then, we have

$$L_q\left[Sin_q(x)\right]=\sum_{n=0}^{\infty}\frac{a_n}{s^{n+1}}=\sum_{m=0}^{\infty}\frac{(-1)^m}{s^{2m+2}}=\frac{1}{1+s^2}\quad(s>1).$$

Similarly, for a sequence $b_n=\begin{cases}(-1)^m, & n=2m\\ 0, & n=2m+1\end{cases}$ $(m=0,\ 1,\ 2,\ \cdots)$, we put

$$Cos_q(x)=\sum_{n=0}^{\infty}\frac{b_n}{(n!)}x^n=\sum_{m=0}^{\infty}\frac{(-1)^m}{\{(2m)!\}_q}x^{2m}\quad(0\le x<\infty_q).$$

Then, we have

$$L_q\left[Cos_q(x)\right]=\sum_{n=0}^{\infty}\frac{b_n}{s^{n+1}}=\sum_{m=0}^{\infty}\frac{(-1)^m}{s^{2m+1}}=\frac{s}{1+s^2}\quad(s>1).$$

And, From (DE3), we have $F_q(s)=\dfrac{As+B}{s^2+1}$. Therefore $f(x)=ACos_q(x)+BSin_q(x)$ is a soluton of (DE3).

APPENDIX

In (3), a q-analogue of error function is defined by

$$Erf_q(x\ ;\alpha)=\frac{1}{\Gamma_q(\alpha)}\int_0^x t^{\alpha-1}E_q(-qt)\ d_qt\quad(0<\alpha<1),$$

and we obtain that the equation

$$e_q(x)Erf_q(x\ ;\alpha)=D_q^{-\alpha}e_q(x)$$

is satisfied. Where a non-integral order differntial operator $D_q^{-\alpha}$ is defined by

$$D_q^{-\alpha}f(x)=(1-q)^{\alpha}x^{\alpha}\sum_{n=0}^{\infty}\frac{(q^{\alpha}\ ;\ q)_n}{(q\ ;\ q)_n}q^n f(q^n x).$$

That is, we have

$$\begin{aligned}e_q(x)Erf_q(x\ ;\alpha)&=(1-q)^{\alpha}x^{\alpha}\sum_{n=0}^{\infty}\frac{(q^{\alpha}\ ;\ q)_n}{(q\ ;\ q)_n}q^n e_q(q^n x)\\&=\frac{(1-q)^{\alpha-1}(q^{\alpha}\ ;\ q)_{\infty}x^{\alpha-1}}{(q\ ;\ q)_{\infty}}\sum_{n=0}^{\infty}(1-q)q^n x\frac{(q^{1+n}\ ;\ q)_n}{(q^{\alpha+n}\ ;\ q)_n}e_q(q^n x)\\&=\frac{1}{\Gamma_q(\alpha)}\int_0^x x^{\alpha-1}\frac{(qt/x\ ;\ q)_{\infty}}{(q^{\alpha}t/x\ ;\ q)_{\infty}}\ e_q(t)\ d_qt\\&=\frac{1}{\Gamma_q(\alpha)}\int_0^x(x-[qt])^{\alpha-1}\ e_q(t)\ d_qt\end{aligned}$$

On the other hand, from

$$L_q[x^{\alpha}e_q(\lambda x)]=\sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)_q}L_q[x^{\alpha+n}]=\sum_{n=0}^{\infty}\frac{\lambda^n\ \Gamma_q(\alpha+n+1)}{(n!)_q s^{\alpha+n+1}},$$

we have

$$L_q[e_q(x)Erf_q(x\ ;\alpha)]=L_q\left[(1-q)^{\alpha}x^{\alpha}\sum_{n=0}^{\infty}\frac{(q^{\alpha}\ ;\ q)_n}{(q\ ;\ q)_n}q^n e_q(q^n x)\right]$$

$$
\begin{aligned}
&= (1-q)^{\alpha} \sum_{n=0}^{\infty} \frac{(q^{\alpha}\ ;\ q)_n}{(q\ ;\ q)_n} q^n \sum_{k=0}^{\infty} \frac{q^{nk}\Gamma_q(\alpha+k+1)}{(k!)_n s^{\alpha+k+1}} \\
&= \frac{(1-q)^{\alpha}}{s^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma_q(\alpha+k+1)}{(k!)_q s^k} \sum_{n=0}^{\infty} \frac{(q^{\alpha}\ ;\ q)_n}{(q\ ;\ q)_n} q^{n(k+1)} \\
&= \frac{(1-q)^{\alpha}}{s^{\alpha+1}} \sum_{k=0}^{\infty} \frac{(1-q)^k}{(q\ ;\ q)_k s^k} \cdot \frac{(q\ ;\ q)_{\infty}}{(q^{\alpha+k+1}\ ;\ q)_{\infty}(1-q)^{\alpha+k}} \cdot \frac{(q^{\alpha+k+1}\ ;\ q)_{\infty}}{(q^{k+1}\ ;\ q)_{\infty}} \\
&= \frac{1}{s^{\alpha+1}} \sum_{k=0}^{\infty} \frac{1}{s^k} = \frac{1}{s^{\alpha}(s-1)}
\end{aligned}
$$

We remark that $L_q[x^{\alpha-1}] = \dfrac{\Gamma_q(\alpha)}{s^{\alpha}}$ and $L_q[e_q(s)] = \dfrac{1}{s-1}$. Thus the equation

$$
L_q\left[D^{-\alpha} e_q(x)\right] = \frac{L_q[x^{\alpha-1}]L_q[e_q(x)]}{\Gamma_q(\alpha)}
$$

is satisfied.

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